

EXTREMES OF $\alpha(t)$ -LOCALLY STATIONARY GAUSSIAN PROCESSES WITH NON-CONSTANT VARIANCES

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Abstract: With motivation from [11], in this paper we derive the exact tail asymptotics of $\alpha(t)$ -locally stationary Gaussian processes with non-constant variance functions. We show that some certain variance functions lead to qualitatively new results.

Key Words: Fractional Brownian motion; $\alpha(t)$ -locally stationary; Pickands constants; Gaussian process.

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1. INTRODUCTION AND MAIN RESULT

For $X(t)$, $t \in [0, T]$, $T > 0$ a centered stationary Gaussian process with unit variance and continuous sample paths Pickands derived in [20] that

$$(1) \quad \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim T \mathcal{H}_\alpha a^{1/\alpha} u^{2/\alpha} \mathbb{P} \{X(0) > u\}, \quad u \rightarrow \infty,$$

provided that the correlation function r satisfies

$$(2) \quad 1 - r(t) \sim a |t|^\alpha, \quad t \downarrow 0, \quad a > 0, \quad \text{and } r(t) < 1, \quad \forall t \neq 0,$$

with $\alpha \in (0, 2]$ (\sim means asymptotic equivalence when the argument tends to 0 or ∞). Here the classical Pickands constant \mathcal{H}_α is defined by

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} \left\{ \sup_{t \in [0, T]} e^{\sqrt{2} B_\alpha(t) - t^\alpha} \right\},$$

where $B_\alpha(t)$, $t \geq 0$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, see [20, 21, 8, 13, 7, 14, 9, 23, 10, 12, 5, 15] for various properties of \mathcal{H}_α .

The deep contribution [3] introduced the class of locally stationary Gaussian processes with index α , i.e., a centered Gaussian process $X(t)$, $t \in [0, T]$ with a constant variance function, say equal to 1, and correlation function satisfying

$$r(t, t+h) = 1 - a(t)|h|^\alpha + o(|t|^\alpha), \quad h \rightarrow 0,$$

uniformly with respect to $t \in [0, T]$, where $\alpha \in (0, 2]$ and $a(t)$ is a bounded, strictly positive and continuous function.

Clearly, the class of locally stationary Gaussian processes includes the stationary ones. It allows for some minor fluctuations of dependence at t and at the same time keeps stationary structure at the local scale. See [3, 4, 18] for studies on the locally stationary Gaussian processes with index α .

In [11] the tail asymptotics of the supremum of $\alpha(t)$ -locally stationary Gaussian processes are investigated. Such processes and random fields are of interest in various applications, see [11] and the recent contributions [2, 16, 17]. Following the definition in [11], a centered Gaussian process $X(t)$, $t \in [0, T]$ with continuous sample paths and unit variance is $\alpha(t)$ -locally stationary if the correlation function $r(\cdot, \cdot)$ satisfies the following conditions:

- (i) $\alpha(t) \in C([0, T])$ and $\alpha(t) \in (0, 2]$ for all $t \in [0, T]$;
- (ii) $a(t) \in C([0, T])$ and $0 < \inf\{a(t) : t \in [0, T]\} \leq \sup\{a(t) : t \in [0, T]\} < \infty$;

(iii) uniformly for $t \in [0, T]$

$$1 - r(t, t+h) = a(t)|h|^{\alpha(t)} + o(|h|^{\alpha(t)}), \quad h \rightarrow 0,$$

where $f(t) \in C(\mathcal{T})$ means that $f(t)$ is continuous on $\mathcal{T} \subset \mathbb{R}$.

In this paper, we shall consider the case that the variance function $\sigma^2(t) = \text{Var}(X(t))$ is not constant, assuming instead that:

(iv) $\sigma(t)$ attains its maximum equal to 1 over $[0, T]$ at the unique point $t_0 \in [0, T]$ and for some constants $c, \gamma > 0$,

$$\frac{1}{\sigma(t)} = 1 + ce^{-|t-t_0|^{-\gamma}}(1 + o(1)), \quad t \rightarrow t_0.$$

A crucial assumption in our result is that similar to the variance function, the function $\alpha(t)$ has a certain behaviour around the extreme point t_0 . Specifically, as in [11] we shall assume:

(v) there exist $\beta, \delta, b > 0$ such that

$$\alpha(t + t_0) = \alpha(t_0) + b|t|^\beta + o(|t|^{\beta+\delta}), \quad t \rightarrow 0.$$

Remark 1.1. We remark that t_0 does not need to be the unique point such that $\alpha(t)$ is minimal on $[0, T]$, which is different from [11]. For instance, $[0, T] = [0, 2\pi]$, $t_0 = 0$ and $\alpha(t) = 1 + \frac{1}{2}\sin(t)$, then 0 is not the minimum point of $\alpha(t)$ over $[0, 2\pi]$ which means assumptions about $\alpha(t)$ in [11] are not satisfied but assumption (v) here is satisfied with

$$\alpha(t) = 1 + \frac{1}{2}|t| + o(|t|^{\frac{3}{2}}), \quad t \rightarrow 0.$$

Below we set $\alpha := \alpha(t_0)$, $a := a(t_0)$ and write Ψ for the survival function of an $N(0, 1)$ random variable. Further, define $0^a = \infty$ for $a < 0$. Our main result is stated in the next theorem.

Theorem 1.2. If a centered Gaussian process $X(t), t \in [0, T]$ with continuous sample paths is such that the assumptions (i)-(v) are valid, then we have as $u \rightarrow \infty$

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \widehat{I} a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-\frac{1}{\gamma \wedge \beta}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^{2^{-1/\gamma}} e^{\frac{-2bx^\beta}{\alpha^2}} dx, & \text{if } \gamma = \beta, \\ \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx, & \text{if } \gamma > \beta, \end{cases}$$

where $\gamma \wedge \beta = \min(\gamma, \beta)$ and

$$\widehat{I} = \begin{cases} 1, & \text{if } t_0 = 0 \text{ or } t_0 = T, \\ 2, & \text{if } t_0 \in (0, T). \end{cases}$$

Remark 1.3. i) If $\alpha(t) \equiv \alpha$ for all t in a small neighborhood of t_0 , the asymptotic of $\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\}$ is the same as in the case of $\gamma < \beta$ in Theorem 1.2.

ii) The result of case $\gamma > \beta$ in Theorem 1.2 is the same as the $\alpha(t)$ -locally stationary scenario in [11], which means that $\sigma(t)$ varies so slow in a small neighborhood of t_0 that $X(t)$ can be considered as $\alpha(t)$ -locally stationary in this small neighborhood.

The following example is a straightforward application of Theorem 1.2.

Example 1.4. Here we consider a multifractional Brownian motion $B_{H(t)}(t)$, $t \geq 0$, i.e., a centered Gaussian process with covariance function

$$\mathbb{E} \{ B_{H(t)}(t) B_{H(s)}(s) \} = \frac{1}{2} D(H(s) + H(t)) \left[|s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t-s|^{H(s)+H(t)} \right],$$

where $D(x) = \frac{2\pi}{\Gamma(x+1)\sin(\frac{\pi x}{2})}$ and $H(t)$ is a Hölder function of exponent λ such that $0 < H(t) < \min(1, \lambda)$ for $t \in [0, \infty)$. For constants T_1, T_2 with $0 < T_1 < T_2$, define

$$\overline{B}_{H(t)}(t) := \frac{B_{H(t)}(t)}{\sqrt{\text{Var}(B_{H(t)}(t))}}, \quad t \in [T_1, T_2],$$

and

$$\sigma(t) := 1 - e^{-|t-t_0|^{-\gamma}}, \quad t \in [T_1, T_2],$$

with some $t_0 \in (T_1, T_2)$ and $\gamma > 0$.

By [11], $\overline{B}_{H(t)}(t)$, $t \in [T_1, T_2]$, is a $2H(t)$ -locally stationary Gaussian process with correlation function

$$r(t, t+h) = 1 - \frac{1}{2}t^{-2H(t)}|h|^{2H(t)} + o(|h|^{2H(t)}), \quad h \rightarrow 0.$$

Further, we assume that there exist $\beta, \delta, b > 0$ such that $H(t+t_0) = H(t_0) + bt^\beta + o(t^{\beta+\delta})$, as $t \rightarrow 0$. Then

$$\mathbb{P} \left\{ \sup_{t \in [T_1, T_2]} \sigma(t) \overline{B}_{H(t)}(t) > u \right\} \sim 2^{1-1/2H} \frac{\mathcal{H}_{2H}}{t_0} u^{1/H} (\ln u)^{-\frac{1}{\gamma \wedge \beta}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^{2^{-1/\gamma}} e^{-\frac{bx^\beta}{H^2}} dx, & \text{if } \gamma = \beta, \\ \int_0^\infty e^{-\frac{bx^\beta}{H^2}} dx, & \text{if } \gamma > \beta, \end{cases} \quad u \rightarrow 0.$$

with $H := H(t_0)$.

2. PROOFS

In the rest of the paper, we focus on the case when $t_0 = 0$. The complementary scenario when $t_0 \in (0, T]$ follows by analogous argumentation. Recall that

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{H}_\alpha[0, T], \quad \text{with } \mathcal{H}_\alpha[-S_1, S_2] = \mathbb{E} \left\{ \sup_{t \in [-S_1, S_2]} e^{\sqrt{2}B_\alpha(t) - |t|^\alpha} \right\} \in (0, \infty),$$

where $S_1, S_2 \in [0, \infty)$ with $\max(S_1, S_2) > 0$ are some constants.

Lemma 2.1. *Under the assumptions of Theorem 1.2 we have*

$$(3) \quad \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\}, \quad u \rightarrow \infty.$$

Moreover, there exists a constant $C > 0$ such that for all sufficiently large u

$$(4) \quad \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} \leq CTu^{2/\alpha} (\ln u)^{-4/3\beta} \Psi(u),$$

where for some constant $q > 1$

$$(5) \quad \delta_1(u) = \left(\frac{1}{2 \ln u - q \ln \ln u} \right)^{1/\gamma} \quad \text{and} \quad \delta_2(u) = \left(\frac{\alpha^2 (\ln \ln u)}{\beta (\ln u)} \right)^{1/\beta}.$$

By (4), in the proof of Theorem 1.2, we derive that, as $u \rightarrow \infty$,

$$(6) \quad \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} = o \left(\mathbb{P} \left\{ \sup_{t \in [0, \delta_2(u)]} X(t) > u \right\} \right).$$

Since $\delta_1(u) \rightarrow 0, \delta_2(u) \rightarrow 0$ as $u \rightarrow \infty$ and $a(t)$ is continuous, without loss of generality, we may assume that $a(t) \equiv a(0) = a$ for $t \in ([0, \delta_1(u)] \cup [0, \delta_2(u)])$. Moreover, by assumption (iv), we know that $\sigma(t) > 0$ for $t \in ([0, \delta_1(u)] \cup [0, \delta_2(u)])$. Below we use notation $\overline{X}(t) = \frac{X(t)}{\sigma(t)}$ for all t such that $\sigma(t)$ is positive.

Proof of Theorem 1.2: First we derive the asymptotic of

$$\pi(u) := \mathbb{P} \left\{ \sup_{t \in \Delta(u)} X(t) > u \right\},$$

as $u \rightarrow \infty$, where $\Delta(u) = [0, \delta(u)]$ and

$$\delta(u) = \begin{cases} \delta_1(u), & \text{if } \gamma \leq \beta, \\ \delta_2(u), & \text{if } \gamma > \beta, \end{cases}$$

with $\delta_1(u)$ and $\delta_2(u)$ in (5), which combined with Lemma 2.1 finally shows that

$$(7) \quad \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \pi(u).$$

In the following \mathbb{Q}_i , $i \in \mathbb{N}$, are some positive constants. For some $S > 0$, let $Y_{\nu, u}(t), t \in [0, S]$ be a family of centered stationary Gaussian processes with

$$\text{Cov}(Y_{\nu, u}(s), Y_{\nu, u}(t)) = 1 - (1 - \nu)au^{-2}|s - t|^{\alpha + 2b\delta^\beta(u)},$$

for $\nu \in (0, 1), u > 0$ such that $\alpha + 2b\delta^\beta(u) \leq 2$ and $s, t \in [0, S]$. Further, let $Z_{\nu, u}(t), t \in [0, S]$ be another family of centered stationary Gaussian processes with

$$\text{Cov}(Z_{\nu, u}(s), Z_{\nu, u}(t)) = 1 - (1 + \nu)au^{-2}|s - t|^\alpha,$$

for $\nu \in (0, 1), u > 0$ and $s, t \in [0, S]$. Due to assumptions (i) and (v), α is strictly smaller than 2, which guarantees that covariance function of $Y_{\nu, u}(t), t \in [0, S]$ and $Z_{\nu, u}(t), t \in [0, S]$ are positive-definite. Hence the introduced families of Gaussian processes exist.

By assumption (iv), for any small $\varepsilon \in (0, 1)$

$$(8) \quad 1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1 + (1 + \varepsilon)ce^{-|t|^{-\gamma}},$$

holds for $t \in [0, \delta(u)]$.

Case 1: $\gamma < \beta$. Set for any $\epsilon \in (0, 1)$ and all u large

$$\begin{aligned} N(0) = N(u, 0) &:= \left\lfloor \frac{\delta_1(u)u^{2/\alpha}}{S} \right\rfloor, \quad N_\epsilon(u) = \left\lfloor (1 - \epsilon) \frac{\delta_1(u)u^{2/\alpha}}{S} \right\rfloor = \left\lfloor \frac{(1 - \epsilon)u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma} S} \right\rfloor, \\ B_j(u) = B_{j,0}(u) &= \left[j \frac{S}{u^{2/\alpha}}, (j+1) \frac{S}{u^{2/\alpha}} \right], \quad j \in \mathbb{N}, \quad \mathcal{G}_u^{\pm \epsilon} = u \left(1 + (1 \pm \epsilon)ce^{-(1-\epsilon)\delta_1(u)^{-\gamma}} \right). \end{aligned}$$

We notice the fact that

$$\Psi(\mathcal{G}_u^{\pm \epsilon}) \sim \Psi(u), \quad u \rightarrow \infty,$$

and

$$(9) \quad I_1(u) \leq \pi(u) \leq I_1(u) + I_2(u),$$

where

$$I_1(u) = \mathbb{P} \left\{ \sup_{t \in [0, (1-\epsilon)\delta_1(u)]} X(t) > u \right\}, \quad I_2(u) = \mathbb{P} \left\{ \sup_{t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]} X(t) > u \right\}.$$

Then by Bonferroni's inequality, (8), Lemma 3.1 with $k = 0$ and Lemma 3.2

$$\begin{aligned} I_1(u) &\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} \\ &\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} \bar{X}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\ &\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in [jS, (j+1)S]} \bar{X}(tu^{-2/\alpha}) > \mathcal{G}_u^{-\epsilon} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{u, \nu}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\
&\sim \sum_{j=0}^{N_\epsilon(u)} \mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right] \Psi(\mathcal{G}_u^{-\epsilon}) \\
&\sim \sum_{j=0}^{N_\epsilon(u)} \mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right] \Psi(u) \\
&\sim (1-\epsilon)u^{2/\alpha}\delta_1(u) \frac{\mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right]}{S} \Psi(u) \\
(10) \quad &\sim (1-\epsilon)((1+\nu)a)^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} \delta_1(u) \Psi(u), \quad u \rightarrow \infty, \quad S \rightarrow \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{j=0}^{N_\epsilon(u)-1} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} \geq \sum_{j=0}^{N_\epsilon(u)-1} \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{u, \nu}(t) > \mathcal{G}_u^{+\epsilon} \right\} \\
(11) \quad &\sim (1-\epsilon)((1-\nu)a)^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} \delta_1(u) \Psi(u), \quad u \rightarrow \infty, \quad S \rightarrow \infty.
\end{aligned}$$

Since

$$(12) \quad I_1(u) \geq \sum_{j=0}^{N_\epsilon(u)-1} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} - \sum_{0 \leq j < k \leq N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_k(u)} X(t) > u \right\},$$

and by [11][Lemma 4.5]

$$\begin{aligned}
&\sum_{0 \leq j < k \leq N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_k(u)} X(t) > u \right\} \leq \sum_{0 \leq j < k \leq N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} \bar{X}(t) > u, \sup_{t \in B_k(u)} \bar{X}(t) > u \right\} \\
(13) \quad &= o\left(u^{2/\alpha} \delta_1(u) \Psi(u)\right), \quad u \rightarrow \infty, \quad S \rightarrow \infty, \quad \epsilon \rightarrow 0.
\end{aligned}$$

Thus inserting (11) and (13) into (12), we have

$$\lim_{u \rightarrow \infty} \frac{I_1(u)(2 \ln u - q \ln \ln u)^{1/\gamma}}{u^{2/\alpha} \Psi(u)} \geq (1-\epsilon)((1-\nu)a)^{1/\alpha} \mathcal{H}_\alpha,$$

which combined with (10) gives that

$$(14) \quad I_1(u) \sim \frac{a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma}} \Psi(u), \quad u \rightarrow \infty, \quad \nu \rightarrow 0, \quad \epsilon \rightarrow 0.$$

By (iii) and (v), we have for all u large

$$\mathbb{E} \{ (\bar{X}(t) - \bar{X}(s))^2 \} = 2 - 2r(s, t) \leq \mathbb{Q}_1 |s - t|^\alpha,$$

uniformly holds for $s, t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]$. By Piterbarg inequality for u large enough, see e.g., [22][Theorem 8.1] or an extension in [6][Lemma 5.1]

$$(15) \quad I_2(u) \leq \mathbb{P} \left\{ \sup_{t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]} \bar{X}(t) > u \right\} \leq \mathbb{Q}_2 \epsilon \delta_1(u) u^{2/\alpha} \Psi(u),$$

which implies

$$\lim_{\epsilon \rightarrow 0} \lim_{u \rightarrow \infty} \frac{I_2(u)(2 \ln u - q \ln \ln u)^{1/\gamma}}{u^{2/\alpha} \Psi(u)} = 0.$$

Combining this equation with (9) and (14), we get

$$\pi(u) \sim \frac{a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma}} \Psi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (2 \ln u)^{-1/\gamma} \Psi(u), \quad u \rightarrow \infty.$$

Case 2: $\gamma = \beta$. Set

$$d_k = d_k(u) := \left(\frac{k}{\ln(u)(\ln \ln(u))^{1/\beta}} \right)^{1/\beta}, \quad A_k = A_k(u) := [d_k, d_{k+1}].$$

Further let $M_\epsilon(u) = \max\{k \in \mathbb{N} : d_k \leq (1 - \epsilon)\delta_1(u)\}$ for some $\epsilon \in (0, 1)$, then $M_\epsilon(u) \rightarrow \infty$, $u \rightarrow \infty$. Clearly

$$\bigcup_{k=0}^{M_\epsilon(u)-1} A_k \subset [0, (1 - \epsilon)\delta_1(u)] \subset \bigcup_{k=0}^{M_\epsilon(u)} A_k.$$

We divide each interval A_k into subintervals of length $S/u^{2/\alpha(d_k)}$, i.e.,

$$B_{j,k} = B_{j,k}(u) := \left[d_k + j \frac{S}{u^{2/\alpha(d_k)}}, d_k + (j+1) \frac{S}{u^{2/\alpha(d_k)}} \right]$$

for $j = 0, 1, \dots, N(k)$, where $N(k) = N(k, u) := \left\lfloor \frac{d_{k+1} - d_k}{S} u^{2/\alpha(d_k)} \right\rfloor$. Notice that

$$\bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k}.$$

We have

$$(16) \quad I_1(u) \leq \pi(u) \leq I_1(u) + I_2(u),$$

where

$$I_1(u) = \mathbb{P} \left\{ \sup_{t \in [0, (1-\epsilon)\delta_1(u)]} X(t) > u \right\}, \quad I_2(u) = \mathbb{P} \left\{ \sup_{t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]} X(t) > u \right\}.$$

Then by Bonferroni's inequality

$$(17) \quad \begin{aligned} I_1(u) &\geq \sum_{k=0}^{M_\epsilon(u)-1} \sum_{j=0}^{N(k)-1} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} - \sum_{\substack{(j,k), (j',k') \in \mathcal{L} \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ &=: J_1(u) - J_2(u), \end{aligned}$$

where $\mathcal{L} = \{(j, k) : 0 \leq k \leq M_\epsilon(u) - 1, 0 \leq j \leq N(k) - 1\}$ and

$$(j, k) \prec (j', k') \text{ iff } (k < k') \vee (k = k' \wedge j < j'),$$

and by (8), Lemma 3.1 and Lemma 3.2

$$\begin{aligned} I_1(u) &\leq \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} \\ &\leq \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \bar{X}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\ &\leq \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{\nu, u}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\ &\sim \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathcal{H}_\alpha \left[0, S((1 + \nu)a)^{1/\alpha} \right] \Psi(\mathcal{G}_u^{-\epsilon}) \\ &\sim \sum_{k=0}^{M_\epsilon(u)} \frac{d_{k+1} - d_k}{S} u^{2/\alpha(d_k)} \mathcal{H}_\alpha \left[0, S((1 + \nu)a)^{1/\alpha} \right] \Psi(u) \\ &= \frac{\mathcal{H}_\alpha \left[0, S((1 + \nu)a)^{1/\alpha} \right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{(\ln u) \left(\frac{2(\alpha - \alpha(d_k))}{\alpha \alpha(d_k)} \right)} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\mathcal{H}_\alpha[0, S((1+\nu)a)^{1/\alpha}]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{\frac{-2(1-\epsilon_1)(\ln u)(b d_k^\beta - d_k^{\beta+\delta})}{\alpha^2}} \\
&\leq \frac{\mathcal{H}_\alpha[0, S((1+\nu)a)^{1/\alpha}]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \\
&\quad \times \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{\frac{-2(1-\epsilon_1)b((\ln u)^{1/\beta} d_k)^\beta}{\alpha^2}} e^{\frac{2(1-\epsilon_1)(\ln u) d_k^{\beta+\delta}}{\alpha^2 M_\epsilon(u)+1}},
\end{aligned}$$

as $u \rightarrow \infty$, where $\epsilon_1 \in (0, 1)$ is a small constant.

Moreover, using that $d_{M_\epsilon(u)} \leq (1-\epsilon)\delta_1(u)$ and $\lim_{u \rightarrow \infty} (\ln u)\delta_1(u)^{\beta+\delta} = 0$, we observe that

$$\lim_{u \rightarrow \infty} e^{\frac{2(1-\epsilon_1)(\ln u) d_{M_\epsilon(u)}^{\beta+\delta}}{\alpha^2 M_\epsilon(u)+1}} = 1.$$

Finally, since

$$\lim_{u \rightarrow \infty} \sup_{k=0, \dots, M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) = 0$$

and

$$\lim_{u \rightarrow \infty} (\ln u)^{1/\beta} d_{M_\epsilon(u)+1} = (1-\epsilon) \left(\frac{1}{2}\right)^{1/\beta},$$

we obtain

$$\lim_{u \rightarrow \infty} \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{\frac{-2(1-\epsilon_1)b((\ln u)^{1/\beta} d_k)^\beta}{\alpha^2}} = \int_0^{(1-\epsilon)(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\epsilon_1)bx^\beta}{\alpha^2}} dx.$$

Thus

$$(18) \quad \lim_{u \rightarrow \infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \leq \frac{\mathcal{H}_\alpha[0, S((1+\nu)a)^{1/\alpha}]}{S} \int_0^{(1-\epsilon)(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\epsilon_1)bx^\beta}{\alpha^2}} dx,$$

and letting $S \rightarrow \infty, \epsilon_1, \nu \rightarrow 0$, and $\epsilon \rightarrow 0$, we get the upper bound. Similarly, we derive that

$$(19) \quad \lim_{\epsilon \rightarrow 0} \lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{J_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2bx^\beta}{\alpha^2}} dx.$$

By [11] [Lemma 4.5]

$$\begin{aligned}
J_2(u) &= \sum_{\substack{(j,k), (j',k') \in \mathcal{L} \\ (j,k) < (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\
&\leq \sum_{\substack{(j,k), (j',k') \in \mathcal{L} \\ (j,k) < (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > u, \sup_{t \in B_{j',k'}} \overline{X}(t) > u \right\} \\
(20) \quad &= o\left(u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u)\right), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.
\end{aligned}$$

Thus inserting (19) and (20) into (17), we get

$$(21) \quad \lim_{\epsilon \rightarrow 0} \lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\epsilon_1)bx^\beta}{\alpha^2}} dx.$$

By (15)

$$(22) \quad \lim_{\epsilon \rightarrow 0} \lim_{u \rightarrow \infty} \frac{I_2(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} = 0.$$

Hence according to (16), (18), (21), and (22), we have

$$\pi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2bx^\beta}{\alpha^2}} dx, \quad u \rightarrow \infty.$$

Case 3: $\gamma > \beta$. We consider $\pi(u) = \mathbb{P} \left\{ \sup_{t \in [0, \delta_2(u)]} X(t) > u \right\}$ with

$$\delta_2(u) = \left(\frac{\alpha^2 (\ln(\ln u))}{\beta (\ln u)} \right)^{1/\beta}.$$

Set for some $\varepsilon > 0$

$$\mathcal{F}_u^{\pm \varepsilon} = u \left(1 + (1 \pm \varepsilon) c e^{-(\delta_2(u))^{-\gamma}} \right), \quad \mathcal{K} = \{t \in [0, T] : \sigma(t) \neq 0\},$$

and we observe that

$$\Psi(\mathcal{F}_u^{\pm \varepsilon}) \sim \Psi(u), \quad u \rightarrow \infty.$$

By [11][Theorem 2.1]

$$\begin{aligned} \pi(u) &\leq \mathbb{P} \left\{ \sup_{t \in [0, \delta_2(u)]} \bar{X}(t) > u \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in \mathcal{K}} \bar{X}(t) > u \right\} \\ (23) \quad &\sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1/\beta} \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx \Psi(u), \quad u \rightarrow \infty. \end{aligned}$$

Let $d_k, A_k, B_{j,k}, N(k)$ be the same as in **Case 2** and $M(u) = \max(k \in \mathbb{N} : d_k \leq \delta_2(u))$. Clearly

$$\bigcup_{k=0}^{M(u)-1} A_k \subset [0, \delta_2(u)] \subset \bigcup_{k=0}^{M(u)} A_k, \quad \bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k},$$

and by Bonferroni's inequality

$$\begin{aligned} \pi(u) &\geq \sum_{k=0}^{M(u)-1} \sum_{j=0}^{N(k)-1} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} - \sum_{\substack{(j,k), (j',k') \in \mathcal{L}' \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ (24) \quad &=: J'_1(u) - J'_2(u), \end{aligned}$$

where $\mathcal{L}' = \{(j, k) : 0 \leq k \leq M(u) - 1, 0 \leq j \leq N(k) - 1\}$.

By (8), Lemma 3.1, Lemma 3.2 and similar argumentation as (19) with $\mathcal{G}_u^{\pm \varepsilon}$ replaced by $\mathcal{F}_u^{\pm \varepsilon}$ and the fact that $(\ln u)^{1/\beta} d_{M(u)+1} \rightarrow \infty, u \rightarrow \infty$, we get

$$(25) \quad \lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{J'_1(u) (\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx.$$

By [11][Lemma 4.5]

$$\begin{aligned} J'_2(u) &= \sum_{\substack{(j,k), (j',k') \in \mathcal{L}' \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ &\leq \sum_{\substack{(j,k), (j',k') \in \mathcal{L}' \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \bar{X}(t) > u, \sup_{t \in B_{j',k'}} \bar{X}(t) > u \right\} \\ (26) \quad &= o \left(u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \right), \quad u \rightarrow \infty. \end{aligned}$$

Hence inserting (25) and (26) into (24), we have

$$\lim_{u \rightarrow \infty} \frac{\pi(u) (\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx,$$

which combined with (23) gives that

$$\pi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx, \quad u \rightarrow \infty.$$

Consequently, according to Lemma 2.1 and

$$\pi(u) \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \leq \pi(u) + \mathbb{P} \left\{ \sup_{t \in [\delta(u), T]} X(t) > u \right\},$$

(7) is proved and all claims follow. \square

3. APPENDIX

In this section we present the proofs of the lemmas used in the proof of Theorem 1.2.

Proof of Lemma 2.1: Below \mathbb{Q}_k , $k = 0, 1, 2, \dots$, are some positive constants.

Step 1: First we prove (3). By the continuity of $\sigma(t)$ in $[0, T]$, for any small enough constant $0 < \theta < 1$

$$\sup_{t \in [\theta, T]} \sigma(t) =: \rho(\theta) < \sigma(t_0) = \sigma(0) = 1.$$

Then by Borell inequality in [1]

$$\mathbb{P} \left\{ \sup_{t \in [\theta, T]} X(t) > u \right\} \leq \exp \left(-\frac{(u - \mathbb{Q}_0)^2}{2\rho^2(\theta)} \right) = o(\Psi(u)),$$

as $u \rightarrow \infty$, where $\mathbb{Q}_0 = \mathbb{E} \left\{ \sup_{t \in [0, T]} X(t) \right\} < \infty$.

By assumption (iv), for any small $\varepsilon \in (0, 1)$, when θ small enough

$$1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1 + (1 + \varepsilon)ce^{-|t|^{-\gamma}},$$

holds for $t \in [0, \theta]$. Then

$$\frac{1}{\sigma(t)} \geq 1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \geq 1 + (1 - \varepsilon)cu^{-2}(\ln u)^q$$

uniformly holds for $t \in [\delta_1(u), \theta]$.

Moreover by assumption (i) and (iii), when θ small enough

$$\begin{aligned} \mathbb{E} \{ (X(t) - X(s))^2 \} &= \mathbb{E} \{ X^2(t) \} + \mathbb{E} \{ X^2(s) \} - 2\mathbb{E} \{ X(t)X(s) \} \\ &\leq 2 - 2(1 - 2a(t)|t - s|^{\alpha(t)}) \\ &\leq \mathbb{Q}_1|t - s|^\varsigma \end{aligned}$$

holds uniformly for $s, t \in [0, \theta]$, where $\mathbb{Q}_1 = \sup_{t \in [0, \theta]} 4a(t)$ and $\varsigma = \inf_{t \in [0, \theta]} \alpha(t) > 0$.

Then by Piterbarg inequality

$$\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), \theta]} X(t) > u \right\} \leq \mathbb{Q}_2 \theta u^{2/\varsigma} \Psi(u[1 + (1 - \varepsilon)cu^{-2}(\ln u)^q]) = o(\Psi(u)), \quad u \rightarrow \infty.$$

Further, since

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\delta_1(u), \theta]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\theta, T]} X(t) > u \right\},$$

and

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \geq \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\} \geq \mathbb{P} \{ X(0) > u \} = \Psi(u),$$

we get

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\}, \quad u \rightarrow \infty.$$

Step 2: Next we prove (4). When $\gamma \leq \beta$, since $\delta_1(u) = o(\delta_2(u))$, as $u \rightarrow \infty$ and by **Step 1**

$$\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\} = o(\Psi(u)), \quad u \rightarrow \infty.$$

Then for u large enough, (4) is obvious.

When $\gamma > \beta$, for u large enough, we have $\delta_2(u) < \delta_1(u)$ and

$$\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\}.$$

By **Step 1**, we know for all u large

$$\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\} \leq \Psi(u),$$

and then we just need to deal with $\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\}$.

Since $\delta_1(u) \rightarrow 0$, $u \rightarrow \infty$, then by assumption (v)

$$\alpha(t) > \alpha + \frac{3}{4}b(\delta_2(u))^\beta$$

holds for all $t \in [\delta_2(u), \delta_1(u)]$ when u large enough.

Let $\eta_u = u^{-2/(\alpha + \frac{3}{4}b(\delta_2(u))^\beta)}$. For sufficiently large u and $s, t \in [\delta_2(u), \delta_1(u)]$, there exists a constant $\mathbb{Q}_3 > 0$ such that

$$1 - r(s, t) \leq 1 - e^{-\mathbb{Q}_3 |s-t|^{\alpha + \frac{3}{4}b(\delta_2(u))^\beta}}.$$

Let $Y_u(t), t \geq 0$ be a family of centered stationary Gaussian processes with correlation functions

$$r_Y(s, t) = e^{\mathbb{Q}_3 |s-t|^{\alpha + \frac{3}{4}b(\delta_2(u))^\beta}}.$$

Then from Slepian's inequality we get for any constant $S > 0$

$$\begin{aligned} \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} &\leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} \frac{X(t)}{\sigma(t)} > u \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} Y_u(t) > u \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_u(t) > u \right\} \\ &\leq \sum_{i=0}^{\lfloor S\eta_u^{-1} \rfloor + 1} \mathbb{P} \left\{ \sup_{t \in [i\eta_u, (i+1)\eta_u]} Y_u(t) > u \right\} \\ &\leq (\lfloor S\eta_u^{-1} \rfloor + 1) \mathbb{P} \left\{ \sup_{t \in [0, \eta_u]} Y_u(t) > u \right\}, \end{aligned}$$

for sufficiently large u . Notice that for each $s, t \in [0, 1]$

$$1 - r_Y(\eta_u t, \eta_u s) = \mathbb{Q}_3 u^{-2} |s - t|^{\alpha + \frac{3}{4}b(\delta_2(u))^\beta} (1 + o(1)) = \mathbb{Q}_3 u^{-2} |s - t|^\alpha (1 + o(1)), \quad u \rightarrow \infty.$$

Hence, from [22][Lemma D.1]

$$\mathbb{P} \left\{ \sup_{t \in [0, \eta_u]} Y_u(t) > u \right\} \sim \mathcal{H}_\alpha[1] \Psi(u),$$

as $u \rightarrow \infty$. Combining this with the fact that

$$\begin{aligned} \eta_u^{-1} &= u^{2/(\alpha + \frac{3}{4}b(\delta_2(u))^\beta)} = u^{2/\alpha} u^{2/(\alpha + \frac{3}{4}b(\delta_2(u))^\beta - 2/\alpha)} = u^{2/\alpha} u^{-\frac{2}{\beta}(\delta_2(u))^\beta / (\alpha + \frac{3}{4}b(\delta_2(u))^\beta)} \\ &= u^{2/\alpha} u^{-\frac{2}{\beta} \frac{\alpha^2 (\ln(\ln u))}{\beta (\ln u)}} / (\alpha + \frac{3}{4}b(\delta_2(u))^\beta) \leq u^{2/\alpha} u^{-\frac{4}{3} \frac{\ln(\ln u)}{\beta (\ln u)}} = u^{2/\alpha} (\ln u)^{-4/(3\beta)}, \end{aligned}$$

we get for some constant \mathbb{Q}_4 and all u large enough

$$\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} \leq \mathbb{Q}_4 S u^{2/\alpha} (\ln u)^{-4/3\beta} \Psi(u).$$

Then the result follows. \square

Lemma 3.1. *Under the notation in the proof of Theorem 1.2, for $(j, k) \in \mathcal{U} = \{(j, k) : 0 \leq k \leq M^*(u), 0 \leq j \leq N(k)\}$ and $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = 1$, there exists u_0 such that for each $u \geq u_0$*

$$\begin{aligned} \underline{1)} \quad & \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > f(u) \right\} \geq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{\nu, u}(t) > f(u) \right\}; \\ \underline{2)} \quad & \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > f(u) \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{\nu, u}(t) > f(u) \right\}, \end{aligned}$$

where

$$M^*(u) = \begin{cases} 0, & \text{if } \gamma < \beta, \\ M_\epsilon(u), & \text{if } \gamma = \beta, \\ M(u), & \text{if } \gamma > \beta. \end{cases}$$

Proof of Lemma 3.1: Since the proofs of scenarios $\gamma < \beta$, $\gamma = \beta$, and $\gamma > \beta$ are similar, we only present the proof of $\gamma = \beta$. Set $X_{j,k,u}(t) = \overline{X} \left(d_k + \frac{jS+t}{u^{2/\alpha(d_k)}} \right)$, then $\sup_{t \in B_{j,k}} \overline{X}(t) \stackrel{d}{=} \sup_{t \in [0, S]} X_{j,k,u}(t)$. It is enough to analyze the supremum of $X_{j,k,u}(t)$.

1) For sufficiently large u and $s, t \in [0, T]$

$$\begin{aligned} 1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) &= 1 - \text{Cov} \left(\overline{X} \left(d_k + \frac{jS+s}{u^{2/\alpha(d_k)}} \right), \overline{X} \left(d_k + \frac{jS+t}{u^{2/\alpha(d_k)}} \right) \right) \\ &\geq (1 - \nu/2)^{1/3} a \left| u^{-2/\alpha(d_k)}(s-t) \right|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))} \\ &= (1 - \nu/2)^{1/3} a u^{-2\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))/\alpha(d_k)} |(s-t)|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))} \\ (27) \quad &= (1 - \nu/2)^{1/3} a \times I_1 \times I_2. \end{aligned}$$

We deal with I_1 and I_2 separately. For sufficiently large u , uniformly with respect to k ,

$$\begin{aligned} I_1 &= u^{-2\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))/\alpha(d_k)} \\ &= u^{-2} u^{2(\alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)))/\alpha(d_k)} \\ &= u^{-2} e^{2(\ln u)(\alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)))/\alpha(d_k)} \\ (28) \quad &\geq u^{-2} (1 - \nu/2)^{1/3}, \end{aligned}$$

where the last inequality follows from the fact that

$$\begin{aligned} (\ln u) \left| \alpha(d_k) - \alpha \left(d_k + u^{-2/\alpha(d_k)}(jS+t) \right) \right| &\leq (\ln u) \left(\left| b(d_k)^\beta - b \left(d_k + u^{-2/\alpha(d_k)}(jS+t) \right) \right| + 2\delta_1^{\beta+\delta}(u) \right) \\ &\leq (\ln u) \left(\frac{b}{(\ln u)(\ln \ln u)^{1/\beta}} + 2\delta_1^{\beta+\delta}(u) \right) \\ &\leq \frac{b}{(\ln \ln u)^{1/\beta}} + 2(\ln u) \left(\frac{1}{2 \ln u - q \ln \ln u} \right)^{\frac{\beta+\delta}{\gamma}} \rightarrow 0, \quad u \rightarrow \infty. \end{aligned}$$

For I_2 , we need to prove that

$$(29) \quad I_2 \geq (1 - \nu/2)^{1/3} |s-t|^{\alpha+2b\delta_1^\beta(u)}.$$

Assumption (v) implies that

$$(30) \quad \alpha \left(d_k + u^{-2/\alpha(d_k)}(jS+t) \right) < \alpha + 2b\delta_1^\beta(u)$$

for each $(j, k) \in \mathcal{U}$. Thus if $|s-t| < 1$, then (29) holds immediately. If $1 \leq |s-t| \leq S$, then by (30)

$$\begin{aligned} I_2 &= |(s-t)|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))} \\ &\geq T^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)) - \alpha - 2b\delta_1^\beta(u)} |s-t|^{\alpha+2b\delta_1^\beta(u)} \\ &\geq T^{-2b\delta_1^\beta(u)} |s-t|^{\alpha+2b\delta_1^\beta(u)} \end{aligned}$$

$$\geq (1 - \nu/2)^{1/3} |s - t|^{\alpha + 2b\delta_1^\beta(u)}$$

for sufficiently large u . The above combined with (27), (28) and (29) gives that for sufficiently large u , uniformly with respect to $(j, k) \in \mathcal{U}$,

$$1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) \geq (1 - \nu/2)au^{-2}|s - t|^{\alpha + 2b\delta_1^\beta(u)} \geq 1 - \text{Cov}(Y_{\nu,u}(s), Y_{\nu,u}(t)).$$

Thus by Slepian's inequality 1) is proved.

2) For all u large

$$\begin{aligned} 1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) &= 1 - \text{Cov}\left(\bar{X}\left(d_k + \frac{jS + s}{u^{2/\alpha(d_k)}}\right), \bar{X}\left(d_k + \frac{jS + t}{u^{2/\alpha(d_k)}}\right)\right) \\ &\leq (1 + \nu)^{1/3} a \left|u^{-2/\alpha(d_k)}(s - t)\right|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS + t))}. \end{aligned}$$

Following the argument analogous to that for the proof of 1), we obtain that for sufficiently large u , uniformly with respect to k , and $s, t \in [0, S]$

$$1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) \leq 1 - \text{Cov}(Z_{\nu,u}(s), Z_{\nu,u}(t)).$$

Again the application of Slepian's inequality completes the proof. \square

Lemma 3.2. For $S > 1$, $\nu \in (0, 1)$, and $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = 1$, as $u \rightarrow \infty$, we have

$$\underline{1)} \mathbb{P}\left\{\sup_{t \in [0, S]} Y_{\nu,u}(t) > f(u)\right\} = \mathcal{H}_\alpha[0, S((1 - \nu)a)^{1/\alpha}] \Psi(f(u))(1 + o(1));$$

$$\underline{2)} \mathbb{P}\left\{\sup_{t \in [0, S]} Z_{\nu,u}(t) > f(u)\right\} = \mathcal{H}_\alpha[0, S((1 + \nu)a)^{1/\alpha}] \Psi(f(u))(1 + o(1)).$$

Proof of Lemma 3.2: We present the proof of 1) and omit the proof of 2) since it follows with similar arguments. Following the definition of $Y_{\nu,u}(t)$, for each $s, t \in [0, S]$

$$\begin{aligned} \lim_{u \rightarrow \infty} f^2(u) \left[1 - \text{Cov}\left(Y_{\nu,u}\left(t(a(1 - \nu))^{-1/\alpha}\right), Y_{\nu,u}\left(s(a(1 - \nu))^{-1/\alpha}\right)\right)\right] \\ = \lim_{u \rightarrow \infty} (a(1 - \nu))^{1 - (\alpha + 2b\delta^\beta(u))/\alpha} |s - t|^{\alpha + 2b\delta^\beta(u)} = |s - t|^\alpha. \end{aligned}$$

Moreover, for all $s, t \in [0, S]$, sufficiently large u and some constant $C > 0$

$$\begin{aligned} f^2(u) \left[1 - \text{Cov}\left(Y_{\nu,u}\left(t(a(1 - \nu))^{-1/\alpha}\right), Y_{\nu,u}\left(s(a(1 - \nu))^{-1/\alpha}\right)\right)\right] \\ \leq (a(1 - \nu))^{1 - (\alpha + 2b\delta^\beta(u))/\alpha} |s - t|^{\alpha + 2b\delta^\beta(u)} \leq CT^{2\alpha} |s - t|^\alpha, \end{aligned}$$

where the last inequality follows from the fact that

$$|s - t|^{\alpha + 2b\delta^\beta(u)} \leq |s - t|^\alpha, \text{ if } |s - t| < 1,$$

and

$$|s - t|^{\alpha + 2b\delta^\beta(u)} \leq T^{2\alpha} \leq T^{2\alpha} |s - t|^\alpha, \text{ if } 1 \leq |s - t| \leq T.$$

Hence, by [19][Lemma 7], we conclude that

$$\begin{aligned} \mathbb{P}\left\{\sup_{t \in [0, S]} Y_{\nu,u}(t) > f(u)\right\} &= \mathbb{P}\left\{\sup_{t \in [0, ((1 - \nu)a)^{1/\alpha} S]} Y_{\nu,u}((a(1 - \nu))^{-1/\alpha} t) > f(u)\right\} \\ &= \mathcal{H}_\alpha[0, ((1 - \nu)a)^{1/\alpha} S] \Psi(f(u))(1 + o(1)), \end{aligned}$$

as $u \rightarrow \infty$. This completes the proof. \square

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REFERENCES

- [1] R.J. Adler and J.E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.
- [2] A. Ayache, N-R. Shieh, and Y. Xiao. Multiparameter multifractional Brownian motion: local nondeterminism and joint continuity of the local times. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(4):1029–1054, 2011.
- [3] S.M. Berman. *Sojourns and extremes of stochastic processes*. The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.
- [4] H.U. Bräker. *High Boundary Excursions of Locally Stationary Gaussian Processes*. Universitat Bern, 1993.
- [5] K. Dębicki, S. Engelke, and E. Hashorva. Generalized Pickands constants and stationary max-stable processes. <http://arxiv.org/pdf/1602.01613.pdf>, 2016.
- [6] K. Dębicki, E. Hashorva, and P. Liu. Ruin probabilities and passage times of γ -reflected Gaussian process with stationary increments. <http://arXiv.org/abs/1511.09234>, 2015.
- [7] K. Dębicki and K.M. Kosiński. On the infimum attained by the reflected fractional Brownian motion. *Extremes*, 17(3):431–446, 2014.
- [8] K. Dębicki. Ruin probability for Gaussian integrated processes. *Stochastic Process. Appl.*, 98(1):151–174, 2002.
- [9] K. Dębicki, E. Hashorva, and L. Ji. Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. *Extremes*, 17(3):411–429, 2014.
- [10] K. Dębicki, E. Hashorva, L. Ji, and K. Taibí. Extremes of vector-valued Gaussian processes: Exact asymptotics. *Stochastic Process. Appl.*, 125(11):4039–4065, 2015.
- [11] K. Dębicki and P. Kisowski. Asymptotics of supremum distribution of $\alpha(t)$ -locally stationary Gaussian processes. *Stochastic Process. Appl.*, 118(11):2022–2037, 2008.
- [12] A. B. Dieker and T. Mikosch. Exact simulation of Brown-Resnick random fields at a finite number of locations. *Extremes*, 18:301–314, 2015.
- [13] A.B. Dieker. Extremes of Gaussian processes over an infinite horizon. *Stochastic Process. Appl.*, 115(2):207–248, 2005.
- [14] A.B. Dieker and B. Yakir. On asymptotic constants in the theory of Gaussian processes. *Bernoulli*, 20(3):1600–1619, 2014.
- [15] E. Hashorva. Representations of max-stable processes via exponential tilting. <https://arxiv.org/abs/1605.03208>, 2016.
- [16] E. Hashorva and L. Ji. Extremes of $\alpha(t)$ -locally stationary Gaussian random fields. *Trans. Amer. Math. Soc.*, 368(1):1–26, 2016.
- [17] E. Hashorva, M. Lifshits, and O. Seleznev. Approximation of a random process with variable smoothness. In *Mathematical statistics and limit theorems*, pages 189–208. Springer, Cham, 2015.
- [18] J. Hüsler. Extreme values and high boundary crossings of locally stationary Gaussian processes. *Ann. Probab.*, 18:1141–1158, 1990.
- [19] J. Hüsler and V. I. Piterbarg. On the ruin probability for physical fractional Brownian motion. *Stochastic Process. Appl.*, 113(2):315–332, 2004.
- [20] J. Pickands, III. Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.*, 145:51–73, 1969.
- [21] V. I. Piterbarg. On the paper by J. Pickands “Upcrossing probabilities for stationary Gaussian processes”. *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, 27(5):25–30, 1972.

- [22] V. I. Piterbarg. *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996.
- [23] V. I. Piterbarg. *Twenty Lectures About Gaussian Processes*. Atlantic Financial Press, London, New York, 2015.

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